A discontinuous Galerkin approach to the elasto-acoustic problem on polytopic grids

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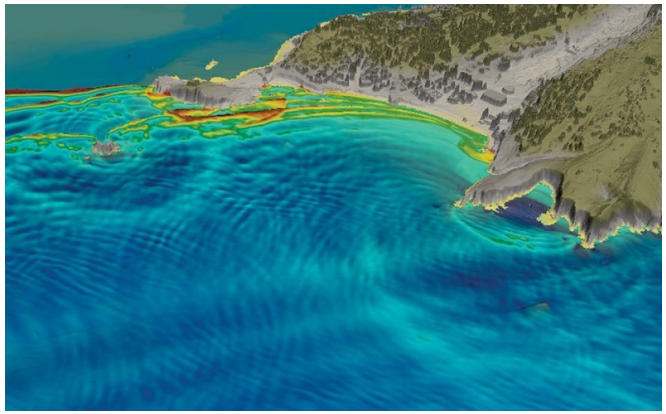
Motivations

- Earthquake scenarios near coastal environments
- Coupling of elastic and acoustic wave propagation

Requirements on the numerical scheme

- Mesh flexibility
- High-order accuracy
- Suited to HPC techniques



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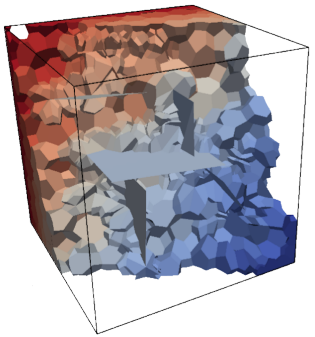
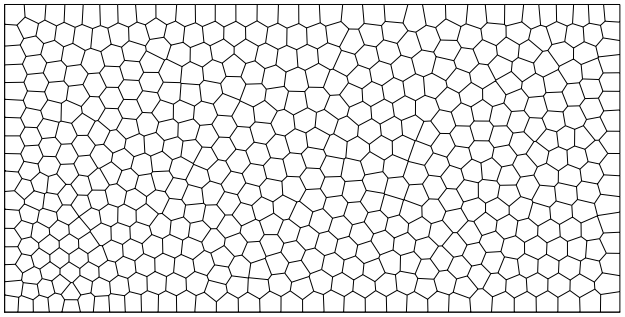
Goal

- Numerical treatment based on polytopic meshes
- The dG method is well-suited to such meshes





State-of-the-art

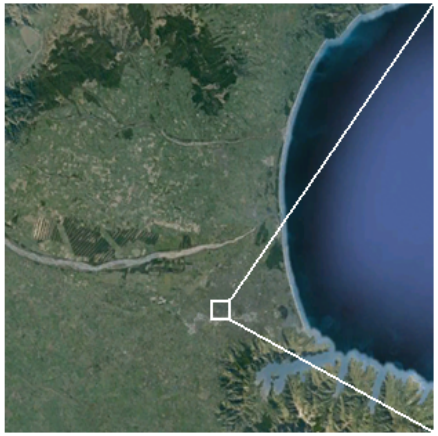
Minimal bibliography

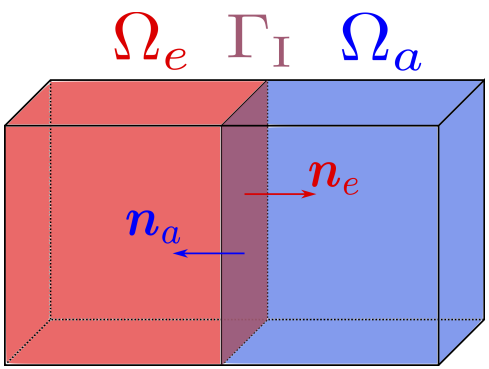
- [Komatitsch *et al.*, 2000]: Spectral Elements
- [Fischer and Gaul, 2005]: FEM–BEM coupling, Lagrange multipliers
- [Flemisch *et al.*, 2006]: classical FEM on two independent meshes
- [Brunner *et al.*, 2009]: FEM–BEM comparison
- [Barucq *et al.*, 2014]: Fréchet differentiability of the elasto-acoustic field
- [Barucq *et al.*, 2014]: dG on simplices, curved edges on interface
- [Péron, 2014]: asymptotic study, equivalent boundary conditions
- [De Basabe and Sen, 2015]: Spectral Elements and Finite Differences
- [Mönköla, 2016]: Spectral Elements, different formulations

Our contribution

- Well-posedness of the coupled problem in the continuous setting
- Detailed analysis of a dG scheme on general polytopic meshes

Elasto-acoustic coupling





Governing equations

$$
\begin{cases}
\rho_e \ddot{\boldsymbol{u}} + 2\rho_e \zeta \dot{\boldsymbol{u}} + \rho_e \zeta^2 \boldsymbol{u} - \mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f}_e & \text{in } \Omega_e \times (0,T], \\
\boldsymbol{\sigma}(\boldsymbol{u}) - \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathbf{0} & \text{in } \Omega_e \times (0,T], \\
\boldsymbol{u} = \mathbf{0} & \text{on } \Gamma_{eD} \times (0,T], \\
\boxed{\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}_e = -\rho_a \dot{\varphi}\, \boldsymbol{n}_e} & \text{on } \Gamma_{\mathrm{I}} \times (0,T], \\
\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1 & \text{in } \Omega_e, \\
c^{-2}\ddot{\varphi} - \triangle \varphi = f_a & \text{in } \Omega_a \times (0,T], \\
\varphi = 0 & \text{on } \Gamma_{aD} \times (0,T], \\
\boxed{\partial\varphi/\partial \boldsymbol{n}_a = -\dot{\boldsymbol{u}} \cdot \boldsymbol{n}_a} & \text{on } \Gamma_{\mathrm{I}} \times (0,T], \\
\varphi(0) = \varphi_0, \quad \dot{\varphi}(0) = \varphi_1 & \text{in } \Omega_a
\end{cases}
$$

- Acoustic pressure exerted by the fluid onto the elastic body
- The normal component of $\boldsymbol{v}_a = -\boldsymbol{\nabla}\varphi$ is continuous at Γ_{I}

Well-posedness

Theorem (Existence and uniqueness)

Under suitable regularity hypotheses on initial data and source terms, there is a **unique strong solution** s.t.

$$\boldsymbol{u} \in C^2([0,T]; \boldsymbol{L}^2(\Omega_e)) \cap C^1([0,T]; \boldsymbol{H}^1_D(\Omega_e)) \cap C^0([0,T]; \boldsymbol{H}^{\triangle}_{\mathbb{C}}(\Omega_e) \cap \boldsymbol{H}^1_D(\Omega_e)),$$
$$\varphi \in C^2([0,T]; L^2(\Omega_a)) \cap C^1([0,T]; H^1_D(\Omega_a)) \cap C^0([0,T]; H^{\triangle}(\Omega_a) \cap H^1_D(\Omega_a))$$

$$\boldsymbol{H}^{\triangle}_{\mathbb{C}}(\Omega_e) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega_e) : \mathbf{div}\, \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{v}) \in \boldsymbol{L}^2(\Omega_e)\},$$
$$H^{\triangle}(\Omega_a) = \{v \in L^2(\Omega_a) : \triangle v \in L^2(\Omega_a)\}$$

Proof. Apply Hille–Yosida upon rewriting the system as

$$\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t}(t) + A\mathcal{U}(t) = \mathcal{F}(t), \quad t \in (0,T],$$
$$\mathcal{U}(0) = \mathcal{U}_0$$

Mesh

Assumptions

- Nonconforming **polytopic** mesh $\mathcal{T}_h = \mathcal{T}_h^e \cup \mathcal{T}_h^a$
- **Arbitrary** number of faces per element:

$$\text{(i) } h_\kappa \lesssim \frac{d|\kappa_\flat^F|}{|F|}, \qquad \text{(ii) } \bigcup_{F \subset \partial\kappa} \overline{\kappa}_\flat^F \subseteq \overline{\kappa}$$

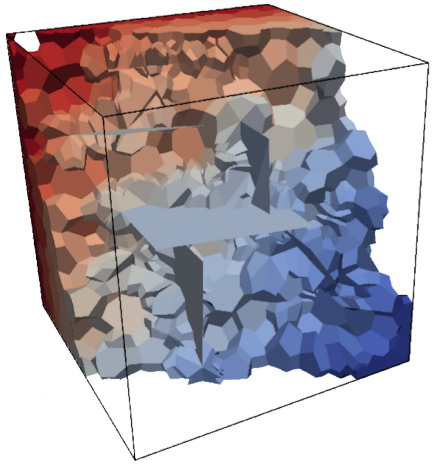
- Possible presence of **degenerating faces**

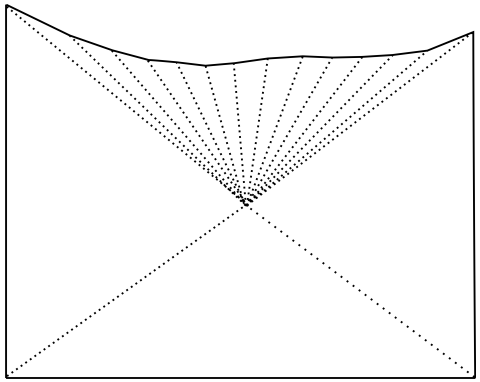
Consequences

- **Discrete trace inequality**:

$$\forall \kappa \in \mathcal{T}_h,\ \forall v \in \mathscr{P}_p(\kappa), \quad \|v\|_{L^2(\partial\kappa)} \lesssim p h_\kappa^{-1/2} \|v\|_{L^2(\kappa)}$$

- **Approximation results** in $\mathscr{P}_p(\kappa)$





♣ [Cangiani *et al.*, '17]

♣ [Antonietti *et al.*, '17]

Semi-discrete problem (SIP dG)

$$\boldsymbol{V}_h^e = \{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega_e) : \boldsymbol{v}_{h|\kappa} \in [\mathscr{P}_{p_{e,\kappa}}(\kappa)]^d,\ p_{e,\kappa} \geqslant 1\ \forall \kappa \in \mathcal{T}_h^e\},$$
$$V_h^a = \{\psi_h \in L^2(\Omega_a) : \psi_{h|\kappa} \in \mathscr{P}_{p_{a,\kappa}}(\kappa),\ p_{a,\kappa} \geqslant 1\ \forall \kappa \in \mathcal{T}_h^a\}$$

Find $(\boldsymbol{u}_h, \varphi_h) \in C^2([0,T]; \boldsymbol{V}_h^e) \times C^2([0,T]; V_h^a)$ s.t., for all $(\boldsymbol{v}_h, \psi_h) \in \boldsymbol{V}_h^e \times V_h^a$,

$$\begin{aligned}(\rho_e \ddot{\boldsymbol{u}}_h(t), \boldsymbol{v}_h)_{\Omega_e} + (c^{-2}\rho_a \ddot{\varphi}_h(t), \psi_h)_{\Omega_a} + (2\rho_e \zeta \dot{\boldsymbol{u}}_h(t), \boldsymbol{v}_h)_{\Omega_e} + (\rho_e \zeta^2 \boldsymbol{u}_h(t), \boldsymbol{v}_h)_{\Omega_e} \\ + \mathcal{A}_h^e(\boldsymbol{u}_h(t), \boldsymbol{v}_h) + \mathcal{A}_h^a(\varphi_h(t), \psi_h) + \mathcal{I}_h^e(\dot{\varphi}_h(t), \boldsymbol{v}_h) + \mathcal{I}_h^a(\dot{\boldsymbol{u}}_h(t), \psi_h) \\ = (\boldsymbol{f}_e(t), \boldsymbol{v}_h)_{\Omega_e} + (\rho_a f_a(t), \psi_h)_{\Omega_a}\end{aligned}$$

$$\begin{aligned}\mathcal{A}_h^e(\boldsymbol{u}, \boldsymbol{v}) &= (\mathbb{C}\boldsymbol{\varepsilon}_h(\boldsymbol{u}), \boldsymbol{\varepsilon}_h(\boldsymbol{v}))_{\Omega_e} - \langle \{\mathbb{C}\boldsymbol{\varepsilon}_h(\boldsymbol{u})\}, [\![\boldsymbol{v}]\!] \rangle_{\mathcal{F}_h^e} \\ &\quad - \langle [\![\boldsymbol{u}]\!], \{\mathbb{C}\boldsymbol{\varepsilon}_h(\boldsymbol{v})\} \rangle_{\mathcal{F}_h^e} + \langle \eta [\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!] \rangle_{\mathcal{F}_h^e} \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_h^e, \\ \mathcal{A}_h^a(\varphi, \psi) &= (\rho_a \boldsymbol{\nabla}_h \varphi, \boldsymbol{\nabla}_h \psi)_{\Omega_a} - \langle \{\rho_a \boldsymbol{\nabla}_h \varphi\}, [\![\psi]\!] \rangle_{\mathcal{F}_h^a} \\ &\quad - \langle [\![\varphi]\!], \{\rho_a \boldsymbol{\nabla}_h \psi\} \rangle_{\mathcal{F}_h^a} + \langle \chi [\![\varphi]\!], [\![\psi]\!] \rangle_{\mathcal{F}_h^a} \qquad \forall \varphi, \psi \in V_h^a, \\ \mathcal{I}_h^e(\psi, \boldsymbol{v}) &= (\rho_a \psi \boldsymbol{n}_e, \boldsymbol{v})_{\Gamma_{\mathrm{I}}} = \langle \rho_a \psi \boldsymbol{n}_e, \boldsymbol{v} \rangle_{\mathcal{F}_{h,\mathrm{I}}} \qquad \forall (\psi, \boldsymbol{v}) \in V_h^a \times \boldsymbol{V}_h^e, \\ \mathcal{I}_h^a(\boldsymbol{v}, \psi) &= (\rho_a \boldsymbol{v} \cdot \boldsymbol{n}_a, \psi)_{\Gamma_{\mathrm{I}}} = -\mathcal{I}_h^e(\psi, \boldsymbol{v}) \qquad \forall (\boldsymbol{v}, \psi) \in \boldsymbol{V}_h^e \times V_h^a\end{aligned}$$

Semi-discrete stability and error estimate

Define the following **energy norm** for $\boldsymbol{W} = (\boldsymbol{v}, \psi) \in C^1([0,T]; \boldsymbol{V}_h^e) \times C^1([0,T]; V_h^a)$:

$$\|\boldsymbol{W}(t)\|_{\mathcal{E}}^2 = \|\rho_e^{1/2}\dot{\boldsymbol{v}}(t)\|_{\Omega_e}^2 + \|\rho_e^{1/2}\zeta\boldsymbol{v}(t)\|_{\Omega_e}^2 + \|\boldsymbol{v}(t)\|_{\mathrm{dG},e}^2 + \|c^{-1}\rho_a^{1/2}\dot{\psi}(t)\|_{\Omega_a}^2 + \|\psi(t)\|_{\mathrm{dG},a}^2$$

Stability

For sufficiently large stabilization parameters,

$$\|(\boldsymbol{u}_h(t), \varphi_h(t))\|_{\mathcal{E}} \lesssim \|(\boldsymbol{u}_h(0), \varphi_h(0))\|_{\mathcal{E}} + \int_0^t (\|\boldsymbol{f}_e(\tau)\|_{\Omega_e} + \|f_a(\tau)\|_{\Omega_a})\,\mathrm{d}\tau$$

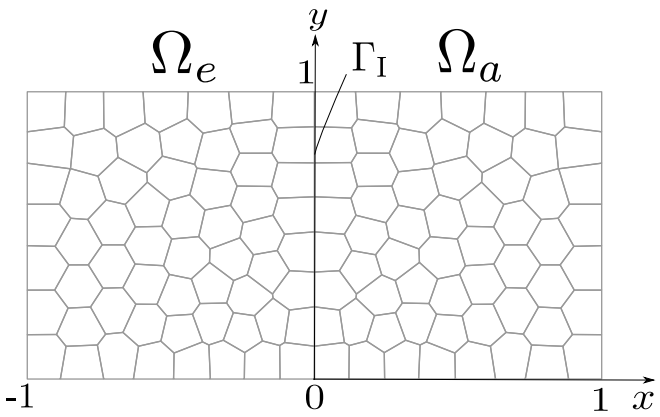
Energy-error estimate

Provided $(\boldsymbol{u}, \varphi) \in C^2([0,T]; \boldsymbol{H}^m(\Omega_e)) \times C^2([0,T]; H^n(\Omega_a))$, $m \geqslant p_e + 1$, $n \geqslant p_a + 1$,

$$\sup_{t\in[0,T]} \|(\boldsymbol{e}_e(t), e_a(t))\|_{\mathcal{E}} \lesssim C_{\boldsymbol{u}}(T)\frac{h^{p_e}}{p_e^{m-3/2}} + C_{\varphi}(T)\frac{h^{p_a}}{p_a^{m-3/2}}$$

Proof. Properly use discrete trace inequality to bound interface contributions

Numerical example I



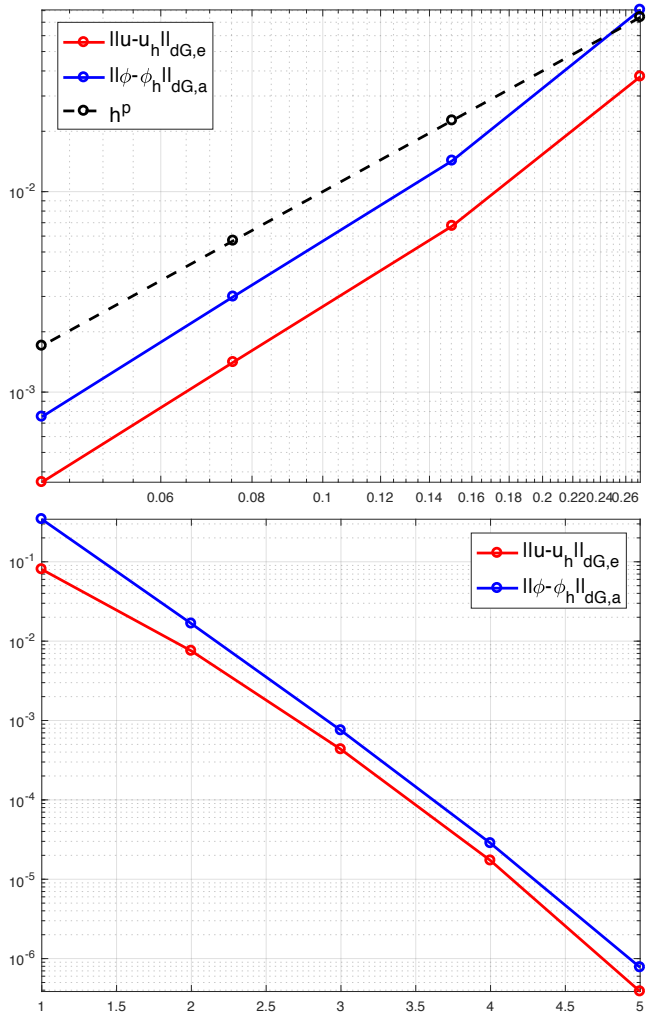
Test case 1

We solve the elasto-acoustic problem on $\Omega_e \cup \Omega_a$, for $T = 1$, $\Delta t = 10^{-4}$, for a homogeneous isotropic elastic material, s.t.

$$\boldsymbol{u}(x, y; t) = x^2 \cos(\sqrt{2}\pi t) \cos\left(\frac{\pi}{2} x\right) \sin(\pi y)\, \widehat{\boldsymbol{u}},$$

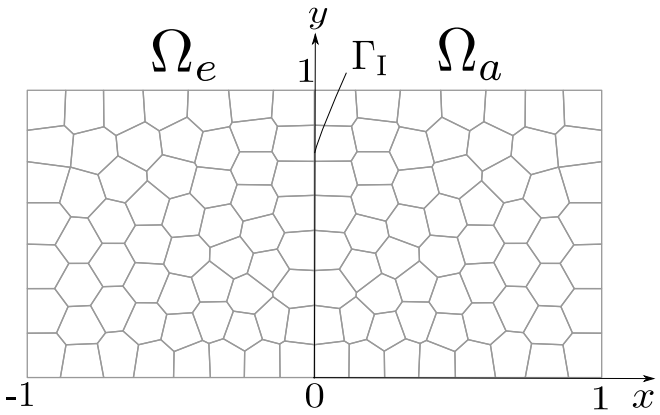
$$\varphi(x, y; t) = x^2 \sin(\sqrt{2}\pi t) \sin(\pi x) \sin(\pi y),$$

with $\widehat{\boldsymbol{u}} = (1, 1)$



$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{dG},e}$ and $\|\varphi - \varphi_h\|_{\mathrm{dG},a}$ vs. h (top) and p (bottom) at $T = 1$

Numerical example II



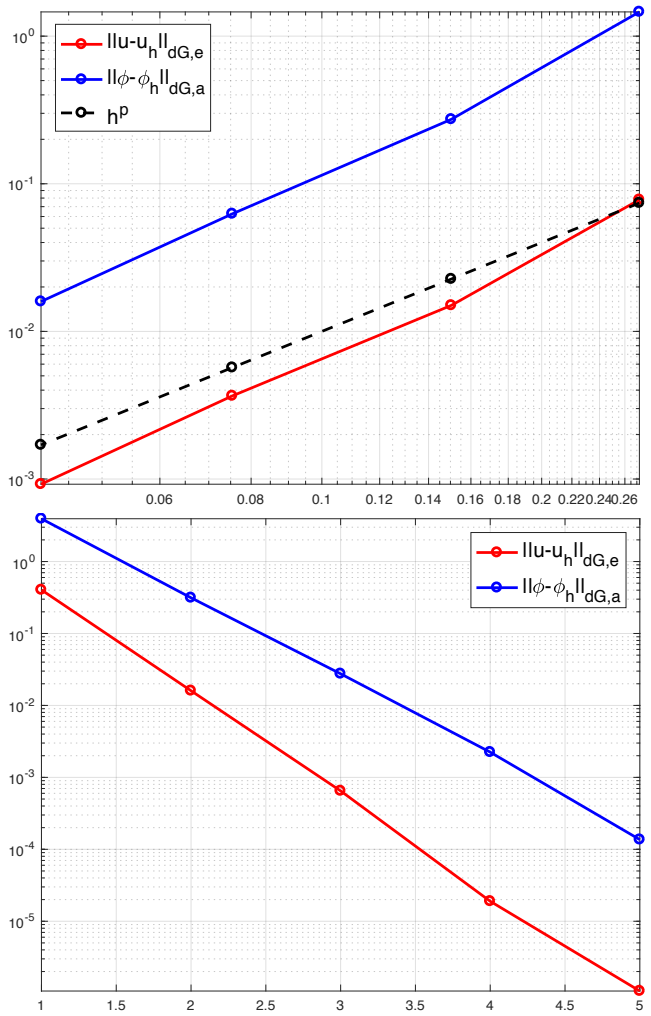
Test case 2 [Mönköla, '16]

We solve the elasto-acoustic problem on $\Omega_e \cup \Omega_a$, for $T = 0.8$, $\Delta t = 10^{-4}$, for a homogeneous isotropic elastic material, s.t.

$$\boldsymbol{u}(x, y; t) = \left(\cos\left(\frac{4\pi x}{c_p}\right), \cos\left(\frac{4\pi x}{c_s}\right)\right)\cos(4\pi t),$$

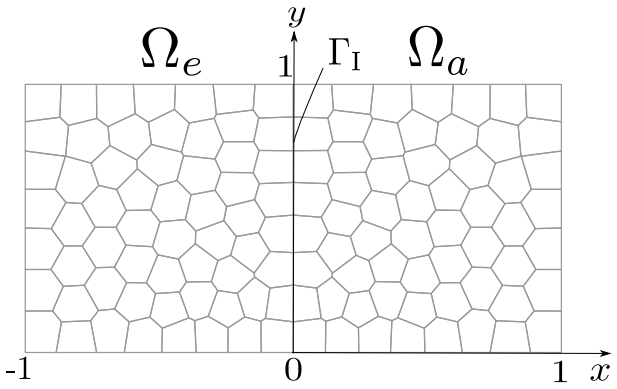
$$\varphi(x, y; t) = \sin(4\pi x)\sin(4\pi t),$$

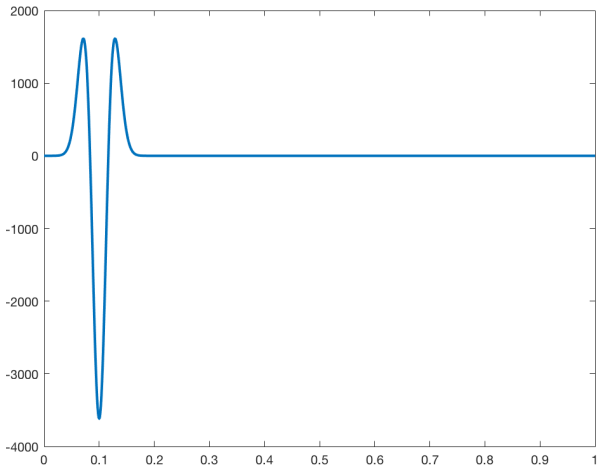
$$c_p = \sqrt{\frac{\lambda + 2\mu}{\rho_e}}, \quad c_s = \sqrt{\frac{\mu}{\rho_e}}$$



$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{dG},e}$ and $\|\varphi - \varphi_h\|_{\mathrm{dG},a}$ vs. h (top) and p (bottom) at $T = 0.8$

Physical example





$t \mapsto \|\boldsymbol{u}(x,y;t)\|_2$ and $t \mapsto |\varphi(x,y;t)|$

Physical example

Point seismic source in the acoustic domain:

$$f_a(\boldsymbol{x},t) = -2\pi\alpha\left(1-2\pi\alpha(t-t_0)^2\right)e^{-\pi\alpha(t-t_0)^2}\delta(\boldsymbol{x}-\boldsymbol{x}_0),$$

$$\boldsymbol{x}_0 \in \Omega_a,\ t_0 \in (0,T],$$

$$\boldsymbol{x}_0 = (0.2, 0.5), \quad t_0 = 0.1$$

Conclusions & perspectives

Conclusions

- We proved that the elasto-acoustic problem is well-posed in the continuous setting
- We proved and validated hp-convergence for a dG method on polytopic meshes
- We used the method to simulate an example of physical interest

Perspectives

- Simulating 3D scenarios, using SPEED (http://speed.mox.polimi.it/)
- Considering the case of totally absorbing boundary conditions
- Inferring error estimates for the fully discrete problem
- Enriching the model by considering a viscoelastic material response:

$$\boldsymbol{\sigma}(\boldsymbol{u}(\boldsymbol{x},t);t) = \mathbb{C}(\boldsymbol{x},0)\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x},t)) - \int_0^t \frac{\partial \mathbb{C}}{\partial s}(\boldsymbol{x},t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x},s))\,\mathrm{d}s$$

References I

P. F. Antonietti, F. Bonaldi, and I. Mazzieri.
A high-order discontinuous Galerkin approach to the elasto-acoustic problem.
Preprint arXiv:1803.01351 [math.NA], submitted, 2018.

A. Cangiani, Z. Dong, E. H. Georgoulis, and P. Houston.
hp-Version discontinuous Galerkin methods on polygonal and polyhedral meshes.
SpringerBriefs in Mathematics, Springer International Publishing, 2017.

P. F. Antonietti, P. Houston, X. Hu, M. Sarti, and M. Verani.
Multigrid algorithms for hp-version interior penalty discontinuous Galerkin methods on polygonal and polyhedral meshes.
Calcolo, 54 (2017), pp. 1169–1198.

S. Mönköla.
On the accuracy and efficiency of transient spectral element models for seismic wave problems.
Adv. Math. Phys., (2016).

J. D. De Basabe and M. K. Sen.
A comparison of finite-difference and spectral-element methods for elastic wave propagation in media with a fluid-solid interface.
Geophysical Journal International, 200 (2015), pp. 278–298.

References II

H. Barucq, R. Djellouli, and E. Estecahandy.
Characterization of the Fréchet derivative of the elasto-acoustic field with respect to Lipschitz domains.
J. Inverse Ill-Posed Probl., 22 (2014), pp. 1–8.

H. Barucq, R. Djellouli, and E. Estecahandy.
Efficient dG-like formulation equipped with curved boundary edges for solving elasto-acoustic scattering problems.
Int. J. Numer. Meth. Engng, 98 (2014), pp. 747–780.

V. Péron.
Equivalent boundary conditions for an elasto-acoustic problem set in a domain with a thin layer.
ESAIM Math. Model. Numer. Anal., 48 (2014), pp. 1431–1449.

B. Flemisch, M. Kaltenbacher, and B. I. Wohlmuth.
Elasto–acoustic and acoustic–acoustic coupling on non-matching grids.
Int. J. Numer. Meth. Engng, 67 (2006), pp. 1791–1810.

D. Komatitsch, C. Barnes, and J. Tromp.
Wave propagation near a fluid-solid interface: a spectral-element approach,
Geophysics, 65 (2000), pp. 623–631.

Thanks for your attention

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Application of Hille–Yosida

Let $\boldsymbol{w} = \dot{\boldsymbol{u}}$, $\phi = \dot{\varphi}$, and $\mathcal{U} = (\boldsymbol{u}, \boldsymbol{w}, \varphi, \phi)$. We introduce

$$\mathbb{H} = \boldsymbol{H}^1_D(\Omega_e) \times \boldsymbol{L}^2(\Omega_e) \times H^1_D(\Omega_a) \times L^2(\Omega_a),$$

with scalar product

$$\begin{aligned}(\mathcal{U}_1, \mathcal{U}_2)_{\mathbb{H}} = {} & (\rho_e \zeta^2 \boldsymbol{u}_1, \boldsymbol{u}_2)_{\Omega_e} + (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}_1), \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_{\Omega_e} \\ & + (\rho_e \boldsymbol{w}_1, \boldsymbol{w}_2)_{\Omega_e} + (\rho_a \boldsymbol{\nabla}\varphi_1, \boldsymbol{\nabla}\varphi_2)_{\Omega_a} + (c^{-2}\rho_a \phi_1, \phi_2)_{\Omega_a}.\end{aligned}$$

Then, we define the operator $A\colon D(A) \subset \mathbb{H} \to \mathbb{H}$ by

$$A\mathcal{U} = \big(-\boldsymbol{w},\ 2\zeta \boldsymbol{w} + \zeta^2 \boldsymbol{u} - \rho_e^{-1} \mathbf{div}\, \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}),\ -\phi,\ -c^2 \triangle \varphi\big) \quad \forall \mathcal{U} \in D(A),$$

$$\begin{aligned}D(A) = \Big\{ \mathcal{U} \in \mathbb{H} : {} & \boldsymbol{u} \in \boldsymbol{H}^{\triangle}_{\mathbb{C}}(\Omega_e),\ \boldsymbol{w} \in \boldsymbol{H}^1_D(\Omega_e),\ \varphi \in H^{\triangle}(\Omega_a),\ \phi \in H^1_D(\Omega_a); \\ & (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \rho_a \phi \boldsymbol{I})\, \boldsymbol{n}_e = \mathbf{0} \text{ on } \Gamma_{\mathrm{I}},\quad (\boldsymbol{\nabla}\varphi + \boldsymbol{w}) \cdot \boldsymbol{n}_a = 0 \text{ on } \Gamma_{\mathrm{I}} \Big\}.\end{aligned}$$

Finally, let $\mathcal{F} = (\mathbf{0}, \rho_e^{-1} \boldsymbol{f}_e, 0, c^2 f_a)$.

For $\mathcal{F} \in C^1([0,T]; \mathbb{H})$ and $\mathcal{U}_0 \in D(A)$,
find $\mathcal{U} \in C^1([0,T]; \mathbb{H}) \cap C^0([0,T]; D(A))$ s.t.

$$\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t}(t) + A\mathcal{U}(t) = \mathcal{F}(t), \quad t \in (0,T],$$

$$\mathcal{U}(0) = \mathcal{U}_0.$$